Adding of Angles When Multiplying Complex Numbers

Mike Jones

(In the Public Domain. No rights reserved.)

Axiom. **i** is a (non-real) number whose square is -1.

Definition: A **pure imaginary number** is a (non-real) number of the form xi, where x is a non-zero real number.

Definition: A **complex number** is a number of the form x + yi, where each of x and y is a real number. (x and y are said to be the **components** of the number.)

The real numbers are said to constitute the **real axis**, and the pure imaginary numbers (along with the origin) are said to constitute the **imaginary axis**.

Every complex number has a **length**, namely, its distance from the origin, and this length is determined (for non-zero complex numbers) by applying the Pythagorean Theorem to its components. (For 0, the length is 0, of course.)

Every (non-zero) complex number also has an **angle**, namely, that angle, measured counter-clockwise from the non-negative portion of the real axis, to the line determined by the origin and the complex number.

When multiplying two complex numbers, the rule is to **multiply their lengths and add their angles**.

You might wonder why it is that we ADD their angles when MULTIPLYING the numbers, instead of multiplying their angles, but we do. (Since angles determined by two rays are unique only up to a multiple of 2π, multiplying them is not possible.)

Let's take an example. Let's pick a complex number and show that its square has twice the angle of the original number.

So, let's pick the number 3 + 7i. Let's call this number P. Let's call its square Q. Then Q = (3 + 7i)x(3 + 7i) = 9 + 42i + 49(-1) = 9 – 49 + 42i = -40 + 42i.
The length of P, by the Pythagorean Theorem, is $\sqrt{(9 + 49)} = \sqrt{58}$, and the length of Q is $\sqrt{(1600 + 1764)} = \sqrt{3364} = \sqrt{(58 \times 58)} = \sqrt{58} \times \sqrt{58}$. The angle of P is arctan(7/3) = 66.8°, and the angle of Q is 180° – arctan(42/40) = 133.6°, which is twice that of P. Thus, the square of P is obtained by squaring its length and doubling its angle.

(end of document)