The Chain Rule applied to log..(xi)

Chain Rule: (f..g)’ = (f’..g)g’

Definition. i is the identity function, defined by the formula i(x)= x.

Example: We will prove that cos’ = -sin, given that sin’ = cos.
Proof: By the Pythagorean Theorem, 2@cos + 2@sin = 1.
This is an abbreviation for (2@i)..cos + (2@i)..sin = 1.
Differentiating both sides, we have:
((2@i)..cos + (2@i)..sin))’ = 1’.
Since the derivative of a constant is 0, 1’ = 0, and we have
((2@i)..cos + (2@i)..sin))’ = 0.
Thus, since the derivative of a sum is the sum of the derivatives,
((2@i)..cos)’ + ((2@i)..sin)’ = 0.
Then, by two applications of the Chain Rule, we have
((2@i)’..cos)cos’ + ((2@i)’..sin)sin’ = 0.
Then, since (2@i)’ = 2i, we have
((2i..cos)cos’ + (2i..sin)sin’ = 0.
That is,
2coscos’ + 2sinsin’ = 0.
Then, since sin’ = cos, we have
2coscos’ + 2sincos = 0.
Cancelling 2cos from both sides gives
cos’ + sin = 0.
Thus,
cos’ = -sin ∎

Theorem 1. If x is a positive number, then (log..(xi))’ = (log’..(xi))x.
Proof: Let f = log, and let g = xi. Then f’ = log’, and g’ = x. Then
(log..(xi))’ = (f..g)’ = (by the Chain Rule) (f’..g)g’ = (log’..(xi))x. ∎

Theorem 2. If x is a positive number, and for each positive number y, f(y) = log(xy), then f’(y) = (log’(xy))x.
Proof: f = log..(xi), and so by Theorem 1, f’ = (log’..(xi))x. Therefore, for each positive number y, f’(y) = (log’(xy))x. ∎

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