First Application of Young's Inequality

Theorem. If each of x and y is a positive real number, and if p and q are a pair of positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then $$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}.$$

<u>Proof</u>: Note that $p > 1$. $\therefore$ $p-1 > 0$. $\therefore$ i^{p-1} is increasing on $[0, \infty)$, and so is $i^{\frac{1}{p-1}}$.

$\therefore$ by Young's Inequality,

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \int_0^{x^{\frac{1}{p}}} i^{p-1} + \int_0^{y^{\frac{1}{q}}} i^{\frac{1}{p-1}}.$$

Note that $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p}{p} - \frac{1}{p} = \frac{p-1}{p}$,

and so $$\frac{1}{p-1} + 1 = \frac{1}{p-1} + \frac{p-1}{p-1} = \frac{p}{p-1} = q.$$

$$\therefore \; x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{i^p}{p}\Big|_0^{x^{\frac{1}{p}}} + \frac{i^q}{q}\Big|_0^{y^{\frac{1}{q}}} = \frac{x}{p} + \frac{y}{q}. \; \blacksquare$$

(end)