The GM-AM Inequality for Two Numbers
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Definition 1. If x and y are real numbers, then their arithmetic mean, denoted by AM(x,y), is defined by the equation $x + y = AM(x,y) + AM(x,y)$. (In other words, replacing each of x and y by this number gives the same sum.)

Theorem 1. If x and y are real numbers, then $AM(x,y) = (x + y)/2$.
Proof: Obvious.

Definition 2. If x and y are positive real numbers, then their geometric mean, denoted by GM(x,y), is defined by the equation $xy = GM(x,y)GM(x,y)$. (In other words, replacing each of x and y by this number gives the same product.)

Theorem 2. If x and y are positive real numbers, then $GM(x,y) = \sqrt{xy}$.
Proof: Obvious.

Reminder: If x is a real number, then $0 \leq x^2$.

The geometric mean of two positive real numbers never exceeds their arithmetic mean, as shown by the following theorem.

Theorem 3. If x and y are positive real numbers, then $GM(x,y) \leq AM(x,y)$.
Proof: Since each of x and y is a positive real number, each of $\sqrt{x}$ and $\sqrt{y}$ is a real number, and therefore $\sqrt{x} - \sqrt{y}$ is a real number, and therefore $0 \leq (\sqrt{x} - \sqrt{y})^2$.
Then $0 \leq (x)^2 - 2\sqrt{x}\sqrt{y} + (y)^2$.
Then $0 \leq x - 2\sqrt{xy} + y$.
Then $2\sqrt{xy} \leq x + y$.
Then $\sqrt{xy} \leq (x + y)/2$.
Then $GM(x,y) \leq AM(x,y)$.
Q.E.D.

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