YOUNG'S INEQUALITY

If f is strictly increasing and continuous, then,

$$bc - af.a \leq \int_a^b f + \int_{f.a}^{c} \hat{f} ,$$

where $\hat{f}$ is the inverse of f.

Proof: Let g.b be defined as $bc - af.a - \int_a^b f$.

Then g'.b = c - f.b.

Then g'.b = 0 if, and only if, b = $\hat{f}$.c.

Since f is increasing, g has a maximum at $\hat{f}$.c. Thus,

$$g.b \leq g.(\hat{f}.c) = (\hat{f}.c)c - af.a - \int_a^{\hat{f}.c} f$$

$$= (\hat{f}.c)f.(\hat{f}.c)) - af.a - \int_a^{\hat{f}.c} f = if\Big|_a^{\hat{f}.c} - \int_a^{\hat{f}.c} f$$

$$= \int_a^{\hat{f}.c} if' = \int_a^{\hat{f}.c} (\hat{f}..f)f' = \int_{f.a}^{f.(\hat{f}.c)} \hat{f} = \int_{f.a}^{c} \hat{f} ,$$

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