The Zero-Derivative Theorem

In order to learn Calculus well, you have to go through it several times. (We will call each go-through a “pass”.) On the first pass, you will assume (take for granted) a lot, either implicitly, or explicitly. Those things that you assume explicitly are called axioms. (And those things that you assume implicitly are called presuppositions.)

One of the things that we will take for granted on the first pass is something that on a later pass will be called the “Zero-Derivative Theorem”, which states that if a function has a derivative of 0, then the function is a constant.

Zero-Derivative Theorem: If $f' = 0$, then there exists a constant c such that $f = c$.
(taken as an axiom on our first pass through Calculus)

The reason that the Zero-Derivative Theorem is important is that it allows us to prove that if two functions have the same derivative, then they differ by at most a constant.

Near-Equality of Functions Theorem: If $f' = g'$, then there exists a constant c such that $f = g + c$.
Proof: $f' = g' \rightarrow f' - g' = 0 \rightarrow (f - g)' = 0 \rightarrow f - g = c \rightarrow f = g + c$. $\blacksquare$

And the reason that this is important is that we often encounter exactly that situation in derivations, for example in deriving the computational formula for Newton’s Law of Cooling from the descriptive formula.

On a later pass through Calculus, the Zero-Derivative Theorem is proved using what is called the “Mean-Value Theorem for Derivatives”, which itself can be taken as an axiom on, say, your second pass.

Mean-Value Theorem for Derivatives: If f is continuous on [a,b] and differentiable on (a,b), then there exists a point c such that $a < c < b$ such that
$f'(c) = (f(b) - f(b))/(b - a)$.
(We won’t prove it here, but you can find the proof on the web.)

Geometrically, The Mean-Value Theorem for Derivatives is highly plausible, in that it is simply saying that there is a tangent to the curve having the same slope as the secant between two given points, in other words, equaling the average slope between the two points.

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